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A Geometric Picture for Fermion Masses

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Abstract

We describe a geometric picture for the pattern of fermion masses of the three generations which is invariant with respect to the renormalization group below the electroweak scale. Moreover, we predict the upper limit for the ratio between the Dirac masses of the μ and τ neutrinos, $(m_{\nu_\mu}/m_{\nu_\tau}) \leq (9.6 \pm 0.6) \times 10^{-3}$.

The Standard Model [1, 2] of strong and electroweak interactions describes accurately all the observed phenomena of elementary particles. In this framework, the fermion masses are essentially free parameters that must be fixed by experiment.

Many efforts have been devoted to find relations among fermion masses, empirically and/or theoretically. For example, a composite model for "fundamental" fermions proposed in [3] predicts the following relation among the masses of the three generations of quarks and leptons

$$m_{f(3)} \sim \sqrt{\frac{m^3_{f(2)}}{m_{f(1)}}} \tag{1}$$

which is experimentally well verified only for charged leptons.

Semi-empirical sum rules among masses can also be found, as the so-called "generation-changing mass-ratio sum rules" [4]

$$\begin{aligned} \sqrt{\frac{m_c}{m_u}} - \sqrt{\frac{m_s}{m_d}} &= \sqrt{\frac{m_\mu}{m_e}} \\ \sqrt{\frac{m_t}{m_c}} - \sqrt{\frac{m_b}{m_s}} &= \sqrt{\frac{m_\tau}{m_\mu}} \end{aligned} \tag{2}$$

which works fairly well for the experimental values given in [5]. Furthermore, the previous formulas may indicate the quarks and leptons in the second and third generations as the excited states of the corresponding ones of the first generation. Other relations among fermion masses, more or less elegant and experimentally verified have been proposed; see for example [6].

Koide [7] has shown that the last experimental data about the tau lepton mass [5] are in excellent agreement with the following lepton mass formula

$$m_e + m_\mu + m_\tau = \frac{2}{3}\left(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}\right)^2 \tag{3}$$

which he deduced in several different models [8].

An intriguing geometric interpretation of the previous formula has been recently proposed by Foot [9]. He considers a three-dimensional Euclidean vector space and the vector $\mathbf{M}$ of components $(\sqrt{m_0}, \sqrt{m_0}, \sqrt{m_0})$ in a monometric, orthogonal, Cartesian reference frame and a vector $\mathbf{L}$, associated to the charged leptons, with components $(\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau})$. Here we generalize the suggestion of Foot which considers $\mathbf{M}=(1,1,1)$. Though the

dimensional parameter m_0 does not enter explicitly in any physical expression which will be studied in the following, it may be viewed, for example, as the fundamental mass vector which generates by means of some unknown mechanism all known mass vectors.

By defining the angle ϑ_{ML} between these two vectors:

$$\cos\vartheta_{ML} = \frac{\mathbf{M}\cdot\mathbf{L}}{|\mathbf{M}|\,|\mathbf{L}|} = \frac{1}{\sqrt{3}}\,\frac{\sqrt{m_e}+\sqrt{m_\mu}+\sqrt{m_\tau}}{\sqrt{m_e+m_\mu+m_\tau}} \tag{4}$$

the Koide's mass formula (3) is reproduced for $\vartheta_{ML} = 45°$. From the equation (4) and the experimental values of the masses of charged leptons [5],

$$\begin{array}{lcllll} m_e & = & 0.51099906 & \pm & 0.00000015 & MeV \\ m_\mu & = & 105.658389 & \pm & 0.000034 & MeV \\ m_\tau & = & 1777.1 & & {}^{+0.4}_{-0.5} & MeV \end{array} \tag{5}$$

one finds

$$\vartheta_{ML} = 45.0003° \pm 0.0012° \ , \tag{6}$$

which shows the excellent agreement with experiments of the mass formula proposed by Koide.

Thus, we propose to extend these considerations to the masses of the quarks. Introducing for the *down* quarks the vector $\mathbf{D}$=($\sqrt{m_d}$, $\sqrt{m_s}$, $\sqrt{m_b}$), and using for the quark masses the numerical values most recently evaluated *via* QCD sum rules [10][1]

$$\begin{array}{lcllll} m_d & = & 8.3 & \pm & 2.9 & MeV \\ m_s & = & 175 & \pm & 25 & MeV \\ m_b & = & 4700 & \pm & 70 & MeV \ , \end{array} \tag{7}$$

the angle formed by the $\mathbf{M}$ and $\mathbf{D}$ is

$$\vartheta_{MD} = 45.6° \pm 0.6° \tag{8}$$

very close to ϑ_{ML}. It is worth noting that $\mathbf{L}$ and $\mathbf{D}$ are not collinear and not coplanar with $\mathbf{M}$. The cosine of the angle formed by the vector $\mathbf{D}\times\mathbf{M}$ and $\mathbf{L}$ is equal to 0.70 ± 0.02 excluding definitely coplanarity.

If one applies the same procedure to the *up* quark masses [10]

$$\begin{array}{lcllll} m_u & = & 3.7 & \pm & 2.8 & MeV \\ m_c & = & 1460 & \pm & 70 & MeV \\ m_t & = & 174 & \pm & 16 & GeV \ , \end{array} \tag{9}$$

[1] Similar results should be obtained using the "experimental" ranges for quark masses reported in [5]. We choose to use the values estimated in [10] for their relatively small errors.

and defines the vector $\mathbf{U}$=($\sqrt{m_u}$, $\sqrt{m_c}$, $\sqrt{m_t}$), one obtains

$$\vartheta_{MU} = 50.9^\circ \pm 0.2^\circ \ . \tag{10}$$

It is easy to verify that, with the quoted values for the charged fermion masses, the vectors $\mathbf{M}$, $\mathbf{L}$, $\mathbf{D}$, $\mathbf{U}$ do not show notable properties of coplanarity among them.

Now assuming for Dirac neutrino masses the same property of *up* quarks, *i.e.* that the vector ($\sqrt{m_{\nu_e}}$, $\sqrt{m_{\nu_\mu}}$, $\sqrt{m_{\nu_\tau}}$) forms an angle of $50.9^\circ \pm 0.2^\circ$ with $\mathbf{M}$, and making the reasonable assumption that m_{ν_e} is much smaller than the others, and therefore to neglect it leads to an angle slightly larger than ϑ_{MN}, one gets:

$$\frac{1}{\sqrt{3}} \frac{\sqrt{m_{\nu_\mu}} + \sqrt{m_{\nu_\tau}}}{\sqrt{m_{\nu_\mu} + m_{\nu_\tau}}} \ \leq \ 0.63 \pm 0.16 \tag{11}$$

and a ratio

$$\frac{m_{\nu_\mu}}{m_{\nu_\tau}} \ \leq \ (9.6 \pm 0.6) \times \ 10^{-3} \tag{12}$$

for the Dirac masses of the two neutrinos.

By fixing the mass of tau neutrino to the cosmologically relevant value, $m_{\nu_\tau} \sim 7\, eV$ [11], we get from (12)

$$m_{\nu_\mu} \leq \ (6.7 \pm 0.4) \times \ 10^{-2}\, eV \, . \tag{13}$$

Note that this upper limit is consistent with the value $2.4 \cdot 10^{-3}\, eV$ [12] which would enable one to solve the solar neutrino problem in terms of MSW mechanism [13].
Now, relaxing the assumption that $m_{\nu_e} = 0$, the equation (11) becomes:

$$\frac{1}{\sqrt{3}} \frac{\sqrt{m_{\nu_e}} + \sqrt{m_{\nu_\mu}} + \sqrt{m_{\nu_\tau}}}{\sqrt{m_{\nu_e} + m_{\nu_\mu} + m_{\nu_\tau}}} \ = \ 0.63 \pm 0.16 \, . \tag{14}$$

In Figure 1 we plot the (1-σ) allowed region for the ratio $y \equiv \sqrt{m_{\nu_\mu}/m_{\nu_\tau}}$ versus $x \equiv \sqrt{m_{\nu_e}/m_{\nu_\tau}}$. The straight line corresponds to the $y = x$ equation; we assume, in fact, for the neutrino masses the same hierarchy of all other fermions (*i.e.* $x \leq y$).

Last, an important feature of our approach to the mass problem is related to the mass-scale independence of the geometric structure outlined above. So far, we have used, for the fermion masses, the values given in [10]; these values are evaluated at the corresponding typical mass scale.[2] Now, the geometric structure and the values of the angles between

[2]For light quarks the renormalization mass scale is fixed to about $1\, GeV$.

mass vectors are unaltered if we choose only one renormalization scale (μ) for the fermion masses. For example, choosing $\mu = 1\, GeV$, one can simply verify, using the results about running masses given in [14], that all the numerical values of the cosine of the angles are unchanged. Moreover, if we restrict ourselves to consider the mass running below the $SU(2) \otimes U(1)$ symmetry-breaking scale, the expression

$$\mathcal{F}(\mu) = \frac{\sqrt{m_1 M_1} + \sqrt{m_2 M_2} + \sqrt{m_3 M_3}}{\sqrt{m_1 + m_2 + m_3}\sqrt{M_1 + M_2 + M_3}}\,, \tag{15}$$

which corresponds to the cosine of the angle between mass vectors, is μ-independent, since the running-mass equation can always be written in the diagonal form

$$\mu \frac{d}{d\mu} m_f(\mu) \;=\; -\gamma\, m_f(\mu)\,. \tag{16}$$

Remarkably, as the energy scale varies, the angles between the vectors of such a space remain unaffected by radiative corrections. This property holds only below the electroweak breaking scale; by contrast, above this scale, the renormalization-group equations for the masses are coupled and nonlinearities occur [15]. The resulting renormalization scale invariance of $\mathcal{F}(\mu)$ breaks down.

In this paper we have proposed a geometric description for all fermion masses, extending the observation made by Foot about charged leptons; furthermore, this framework enables one to constrain neutrino mass ratios. We have shown that this approach is renormalization mass-scale independent, and this property proves its geometric nature.

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References

[1] S.L. Glashow, *Nucl. Phys.*, **22** (1961) 579;
S. Weinberg, *Phys. Rev. Lett.*, **19** (1967) 1264;
A. Salam, “Weak and Electromagnetic interactions”, in Svartholm N., *Elementary Particle Theory*, Stockholm, Almquist and Wiksell, 1968.

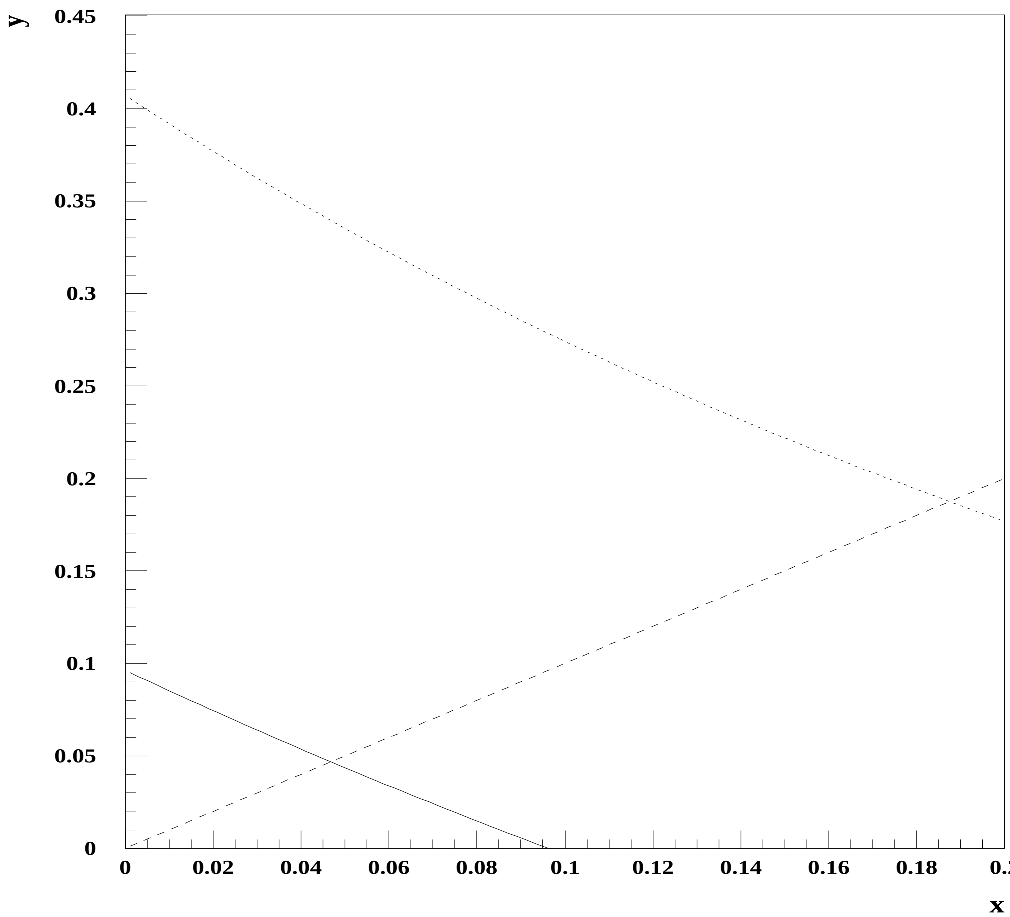

Figure 1: *The allowed (1-σ) range for the ratio* $y = \sqrt{m_{\nu_\mu}/m_{\nu_\tau}}$ *versus* $x = \sqrt{m_{\nu_e}/m_{\nu_\tau}}$ *is reported. The continuous line corresponds to the central value in equation (14), the dotted one to the 1-σ upper limit for $y(x)$. The dashed line is the plot for the equation $y = x$.*

[2] H.Fritzsch, M.Gell-Mann and H.Leutwyler, *Phys. Lett.*, **B47** (1973) 365;
S.Weinberg, *Phys. Rev. Lett.*, **31** (1973) 494;
D.J.Gross and F. Wilczek, *Phys. Rev. Lett.*, **30**,1973,1343;
D.H. Politzer, *Phys. Rev. Lett.*, **30** (1973) 1346.

[3] H.Terazawa and M.Yasuè, *Phys. Lett.* **B307** (1993) 383.

[4] H.Terazawa, *Mod. Phys. Lett.* **A7** (1992) 1879.

[5] Particle Data Group, L. Montanet *et al. Phys. Rev.* **D50** (1994) 1173.

[6] A.Sirlin, *Comm. Nucl. Part. Phys.* **21** (1994) 227;
G.Mangano and F.Sannino, *Mod. Phys. Lett.* **A8** (1993) 1519.

[7] Y. Koide, *Mod. Phys. Lett.* **A8** (1993) 2071.

[8] Y. Koide, *Lett. Nuovo Cim.* **34** (1982) 201, ; *Phys. Lett.* **B120** (1983) 161; *Phys. Rev.* **D28** (1983) 252; *Zeit. für Phys.* **C45** (1989) 39; *Mod. Phys. Lett.* **A5** (1990) 2319.

[9] R. Foot, Preprint McGill/94-9, hep-ph/9402242.

[10] C.A.Dominguez, Preprint UCT-TP-219/94, hep-ph/9410362 ;
J.Bijnens, J.Prades and E. de Rafael, Preprint CPT-94/PE.3097, NORDITA-94/62 N,P, hep-ph/9411285.

[11] V.Berezinsky, Preprint **LNGS-93-85**, invited talk at International Conference on Neutral Currents: Twenty Years Later, Paris, France, 6-9 July 1993.

[12] P. Anselmann *et al.*, *Phys. Lett.*, **B 285** (1992) 309;
S. Esposito, *Mod. Phys. Lett.*, **A 8** (1993) 1557.

[13] L. Wolfenstein, *Phys. Rev.*, **D 17** (1978) 2369;
S.P.Mikheyev and A.Yu.Smirnov, *Il Nuovo Cimento*, **9 C** (1986) 17;
Sov. J. Nucl. Phys., **42** (1986) 913; *Sov. Phys. Usp.*, **30** (1987) 759.

[14] Y. Koide, Preprint **US-94-05**, hep-ph/9410270.

[15] B.Grzadkowski and M.Lindner, *Phys. Lett.* **B193** (1987) 71.